Theoretical aspects of ERa, the fastest practical suffix tree construction algorithm

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Text indexing problem

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Problem statement

Given unstructured input text T consisting of n characters from alphabet Σ build an index such that for query pattern P we:

- determine whether P occurs in T in time $O(P)$,
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Solution

Suffix tree and *suffix array* (SA) with LCP information are fundamental data structures for indexing unstructured text.

Suffix tree construction algorithms

- Theoretical:

	W ('73), McC ('78)	U ('95)	F-C et al. ('00)
Time	$\Theta(n)$	$\Theta(n)$	$\Theta(n \lg \lvert\Sigma\rvert)$
Online	No	Yes	Yes[1]
Cache-efficient	No	No	Yes
Unbounded Σ	No	No	Yes

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	Semi-disk-based			Out-of-core		
	TDD ('04)	ST-Merge ('05)	TRLS. ('07)	B^2ST ('09)	WF ('09)	ERa ('11)
Time	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(n^2)$
Cache-eff.	Yes	Yes	Yes	Yes	Yes	Yes
String acc.	Rnd.	Rnd.	Rnd.	Seq.	Seq.	Seq.
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Motivation

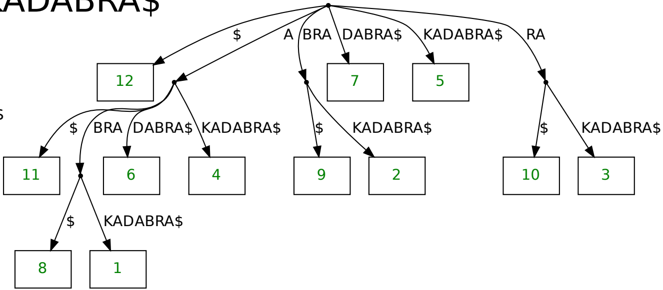
Analyse the fastest practical algorithm and compare it to the theoretical ones.

[1] Bedathur and Haritsa (2004)

Suffix tree

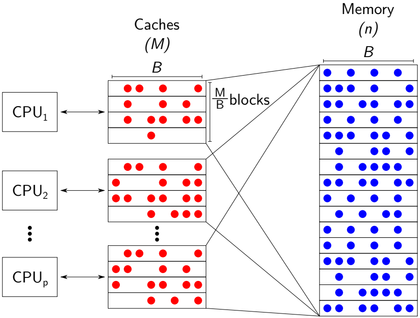
1 2 3 4 5 6 7 8 9 10 11 12
T=ABRAKADABRA$

1 $
2 A$
3 ABRA$
4 ABRAKADABRA$
5 ADABRA$
6 AKADABRA$
7 BRA$
8 BRAKADABRA$
9 DABRA$
10 KADABRA
11 RA$
12 RAKADABRA$



Parallel External Memory model (PEM)[2]

- Shared-memory model describing well the modern multi-core architecture.
- Two-level memory hierarchy w/ fast&limited private caches on 1^{st} level and slow&unlimited main memory on the 2^{nd}.
- Features:
  - P — # of processing elements
  - B — block size
  - M — cache (main memory) size



[2] Arge, Goodrich, Nelson, Sitchinava (2008)

Parallel External Memory model (PEM)

- We assume CREW PEM.
- Data are read and stored on disk and not fitting into main memory:
    - Denote first level as the **main memory** and second level as the **disk**.
- Performance metrics:
    - parallel time,
    - parallel block transfers bw/ disk and main memory.

Suffix tree construction lower bounds

- Sequential:

	bounded Σ	unbounded Σ
Time	$\Omega(Sort(n))$	$\Omega(Sort(n))$
I/Os[3]	$\Omega(Sort(n))$	$\Omega(Sort(n))$
Space[4]	$\Omega(n \lg \lvert\Sigma\rvert)$ bits	$\Omega(n \lg \lvert\Sigma\rvert)$ bits

[3] EM model
[4] Uncompressed index
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ERa - Elastic Range[7]

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- Key idea:
  - DNA, music, proteins are almost random[6]
  - Suppose random input text and PEM model of computation:
    - What is **expected** time complexity?
    - What is **expected** I/O complexity?

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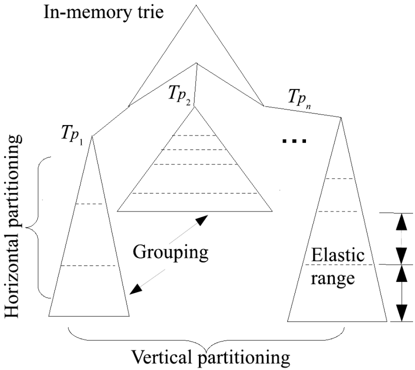
ERa steps

ERa constructs the suffix tree in two steps:

1. The **vertical partitioning** step determines the suffix subtrees just fitting into M.
2. The **horizontal partitioning** step builds the actual suffix subtrees.



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4. To optimally fill the main memory combine the S-prefixes into groups fitting into the main memory as tight as possible.
   - Use First-Fit Decreasing heuristic for bin packing problem[8].

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Vertical partitioning

Analysis:

- For uniform string distributions $O\left(\log_{|\Sigma|} \frac{n}{M}\right)$ scans.
  - Note: The main memory contains $O\left(\frac{n}{M}\right)$ S-prefixes of total size $O(M)$.
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Overall:

- $O\left(n \log_{|\Sigma|} \frac{n}{M} + \left(\frac{n}{M}\right)^2\right)$ time.
- $O\left(\frac{n}{B} \log_{|\Sigma|} \frac{n}{M}\right)$ I/Os.

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5. Construct the suffix subtree using depth-first traversal and reading SA and LCP.

Horizontal partitioning

Analysis:

- Assuming random string distribution, the number of uniquely represented strings in $O(M)$ space is $|\Sigma|^{range}$
  - The number of step two and three iterations is bounded by $O(\log_{|\Sigma|} M)$.
- Each iteration, filling the buffers is done in $O(M)$ time and $O(M/B)$ I/Os.
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Overall for constructing $O\left(\frac{n}{M}\right)$ suffix subtrees:

- $O\left(\frac{n}{p}\log_{|\Sigma|} M\right)$ parallel time.
- $O\left(\frac{n}{pB}\log_{|\Sigma|} M\right)$ parallel block transfers.

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- Future work: Analyse ERa bottlenecks in practice and see if they match the critical terms in time and I/O complexities presented here.

Thank you.

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LUSY
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