Shortest path problem in rectangular complexes of global non-positive curvature

Daniela MAFTULEAC

David R. Cheriton School of Computer Science

University of Waterloo

March 13, 2014

Spaces of non-positive curvature = CAT(0)

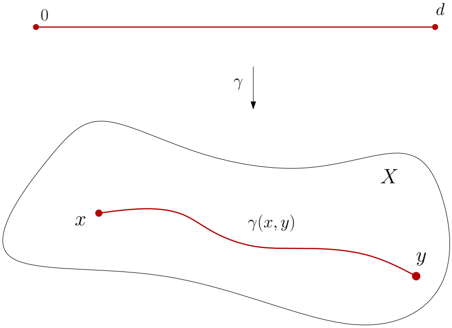
In 1987 M. Gromov introduces the notion of a **CAT(k) space**, in the honour of E. Cartan, A.D. Alexandrov and V.A. Toponogov.

A metric space (X, d) is a **CAT(0) space** if it is geodesically connected and if every geodesic triangle in X is at least as *thin* as its comparison triangle in the Euclidean plane.

Geodesic space or geodesically connected space

Definition

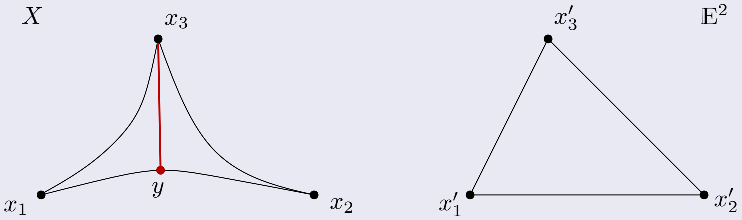
A metric space (X, d) is **geodesic** if for any pair of points x and y of X, such that $d(x, y) = d$, there exists an isometric operator $\gamma : [0, d] \to X$ such that $\gamma(0) = x$, $\gamma(d) = y$ and for all $t, t' \in [0, d]$, $|t - t'| = |\gamma(t) - \gamma(t')|$.



CAT(0) space (global nonpositive curvature)

Definition

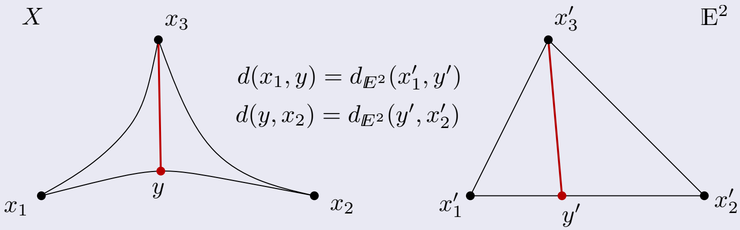
Let (X, d) be a geodesic space and $T = \Delta(x_1, x_2, x_3)$ a geodesic triangle in X. A **comparison triangle** of T is a triangle $T' = \Delta(x_1', x_2', x_3')$ of the Euclidian plane whose edges are of equal length as the edges of T in X.



CAT(0) space (global nonpositive curvature)

Definition

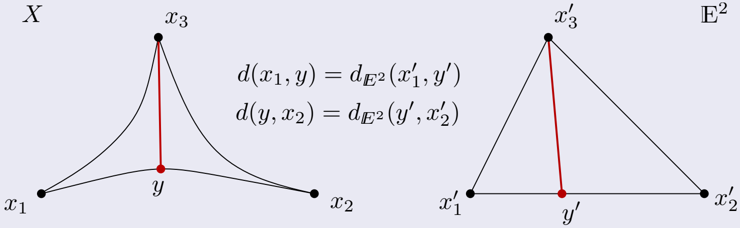
Let (X, d) be a geodesic space and $T = \Delta(x_1, x_2, x_3)$ a geodesic triangle in X. A **comparison triangle** of T is a triangle $T' = \Delta(x_1', x_2', x_3')$ of the Euclidian plane whose edges are of equal length as the edges of T in X.



CAT(0) space (global nonpositive curvature)

Definition

Let (X, d) be a geodesic space and $T = \Delta(x_1, x_2, x_3)$ a geodesic triangle in X. A **comparison triangle** of T is a triangle $T' = \Delta(x_1', x_2', x_3')$ of the Euclidian plane whose edges are of equal length as the edges of T in X.



CAT(0) inequality: $\mathbf{d(x_3, y) \le d_{\mathbb{E}^2}(x_3', y')}$, $\forall y \in \gamma(x_1 x_2)$.

CAT(0) space (global non-positive curvature)

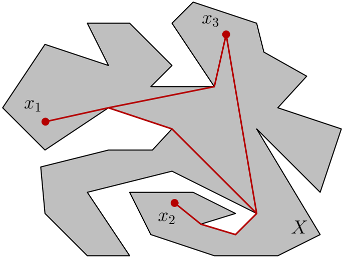
CAT(0) space (M. Gromov, 1987)

A geodesic space (X, d) is a **CAT(0) space** if the CAT(0) inequality is satisfied for every geodesic triangle $T = \Delta(x_1, x_2, x_3)$ of X and every point y of $\gamma(x_1, x_2)$.

CAT(0) space (global non-positive curvature)

CAT(0) space (M. Gromov, 1987)

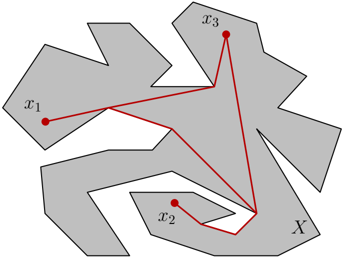
A geodesic space (X, d) is a **CAT(0) space** if the CAT(0) inequality is satisfied for every geodesic triangle $T = \Delta(x_1, x_2, x_3)$ of X and every point y of $\gamma(x_1, x_2)$.



CAT(0) space (global non-positive curvature)

CAT(0) space (M. Gromov, 1987)

A geodesic space (X, d) is a **CAT(0) space** if the CAT(0) inequality is satisfied for every geodesic triangle $T = \Delta(x_1, x_2, x_3)$ of X and every point y of $\gamma(x_1, x_2)$.



Examples of CAT(0) spaces:

- Simple polygons,
- Hyperbolic spaces,
- Trees,
- Euclidian buildings.

CAT(0) space (global non-positive curvature)

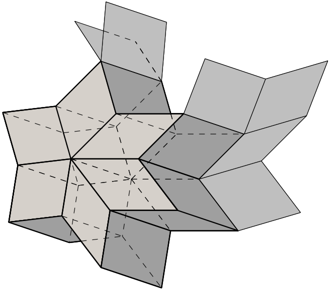
Important properties of CAT(0) spaces:

- **uniqueness of a geodesic connecting two points**
- convexity of the distance function
- global non-positive curvature
- convexity of balls and neighborhoods of convex sets

CAT(0) cube complexes

A **cube complex** $\mathcal{K}$ is a polyhedral complex obtained by gluing solid cubes of various dimensions in such a way that for any two cubes the intersection is a face of both.

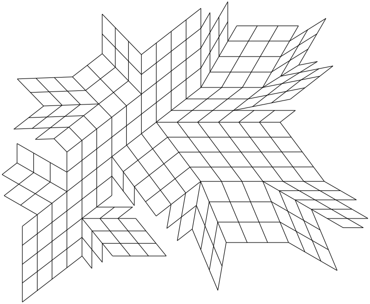
$\mathcal{K}$ has a natural piecewise Euclidean metric.



A cube complex is said to be **CAT(0)** if it is simply-connected and it has non-positive curvature.

CAT(0) cube complexes

2-dimensional CAT(0) cube complex:



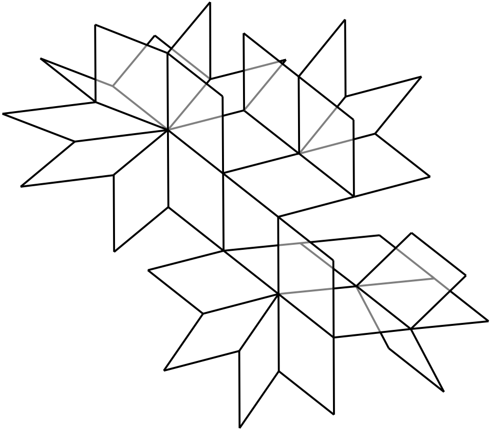
[Daina Taimina, 2005]

CAT(0) cube complexes

[Daina Taimina, 2005]

CAT(0) cube complexes

2-dimensional CAT(0) cube complex:

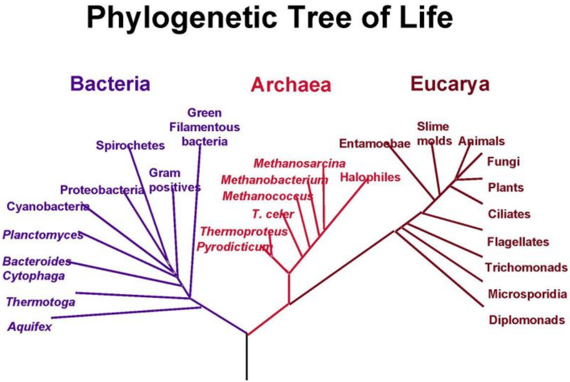


Applications of CAT(0) cube complexes

- Geometry group theory
- Phylogenetic trees
- Reconfigurable systems
- ...

Phylogenetic trees

A **phylogenetic tree** (or evolutionary tree) is a branching diagram showing the inferred evolutionary relationships among various biological species or other entities based upon similarities and differences in their physical and/or genetic characteristics.



Phylogenetic trees

- building trees based on DNA sequences

Phylogenetic trees

- building trees based on DNA sequences
- origins of diseases such as AIDS and the most deadly form of malaria

Phylogenetic trees

- building trees based on DNA sequences
- origins of diseases such as AIDS and the most deadly form of malaria
- connections between tribal groups

Phylogenetic trees

- building trees based on DNA sequences
- origins of diseases such as AIDS and the most deadly form of malaria
- connections between tribal groups
- . . .

Phylogenetic trees

- building trees based on DNA sequences
- origins of diseases such as AIDS and the most deadly form of malaria
- connections between tribal groups
- . . .
- uncertainty remains about precise relationships between the tips or leaves of the tree.

Phylogenetic trees

- building trees based on DNA sequences
- origins of diseases such as AIDS and the most deadly form of malaria
- connections between tribal groups
- . . .
- uncertainty remains about precise relationships between the tips or leaves of the tree.

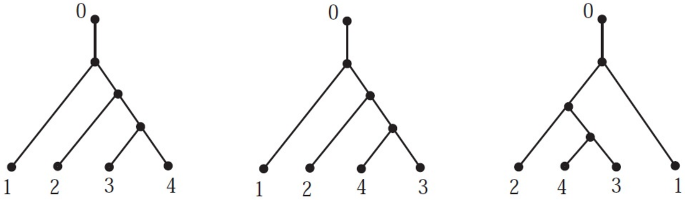
Possible solution: cover all the possible trees with the same set of leaves. There are $(2n-3)!!$ rooted binary trees.

Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)

The space of phylogenetic trees with the same set of leaves and with the intrinsic metric is a CAT(0) cube complex.

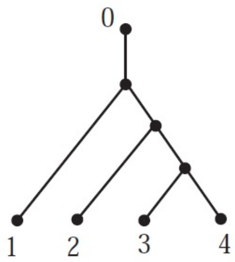
Phylogenetic tree topology ([2001] Billera, Holmes, Vogtmann)

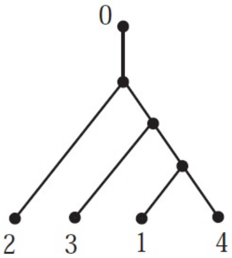
Identical trees



Phylogenetic tree topology ([2001] Billera, Holmes, Vogtmann)

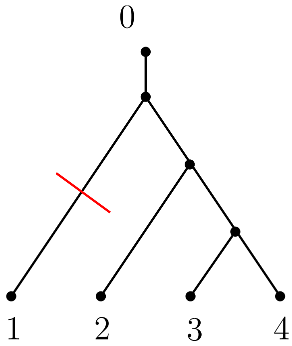
Distinct trees



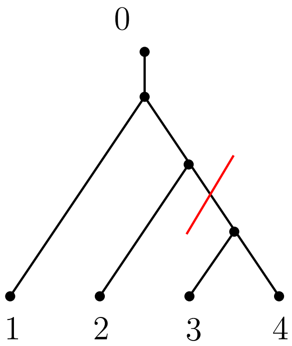


Phylogenetic tree topology ([2001] Billera, Holmes, Vogtmann)

Splits of a phylogenetic tree:



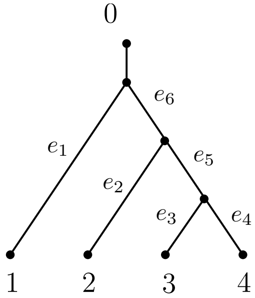
$(1) \mid (0234)$



$(012) \mid (34)$

Phylogenetic tree topology ([2001] Billera, Holmes, Vogtmann)

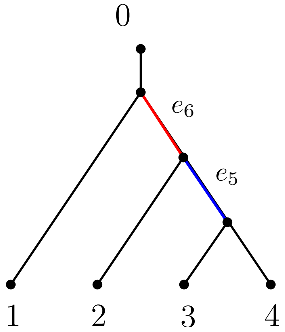
Splits of a phylogenetic tree:



$e_1 : (1) \mid (0234)$	$e_5 : (012) \mid (34)$
$e_2 : (2) \mid (0134)$	$e_6 : (01) \mid (234)$
$e_3 : (3) \mid (0124)$	$e_7 : (013) \mid (24)$
$e_4 : (4) \mid (0123)$	$e_8 : (014) \mid (23)$
	$\dots$

$$\begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ (3; & 2; & 1; & 1; & 1; & 1; & 0; & 0; & ...) \end{matrix}$$

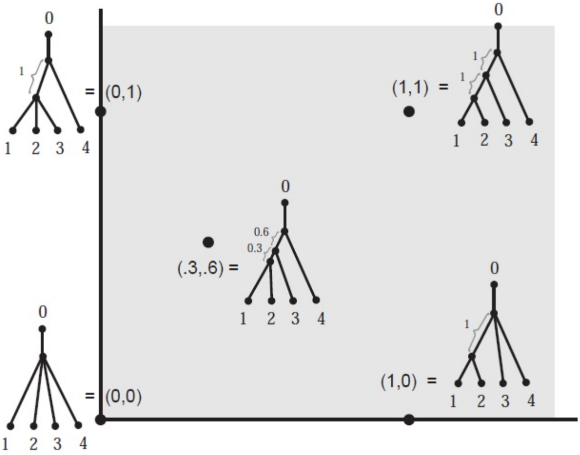
Phylogenetic tree topology ([2001] Billera, Holmes, Vogtmann)



$$\begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & \\ (\cancel{3;} & \cancel{2;} & \cancel{1;} & \cancel{1;} & 1; & 1; & 0; & 0; & \ldots) \end{matrix}$$

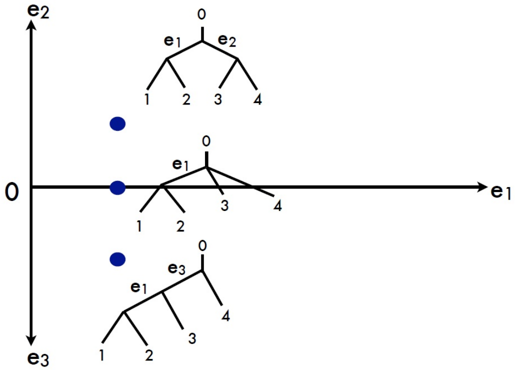
Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)

A 2-dimensional quadrant corresponding to a metric 4-tree $\mathcal{T}_4$

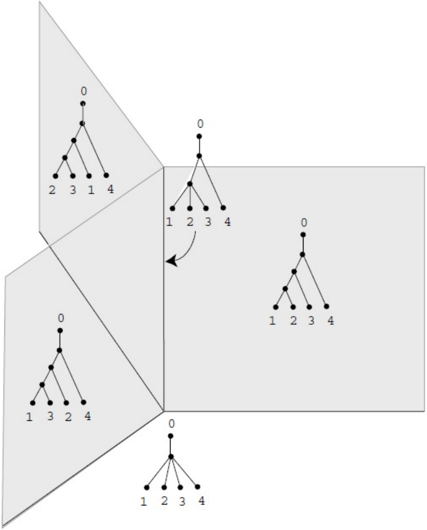


Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)

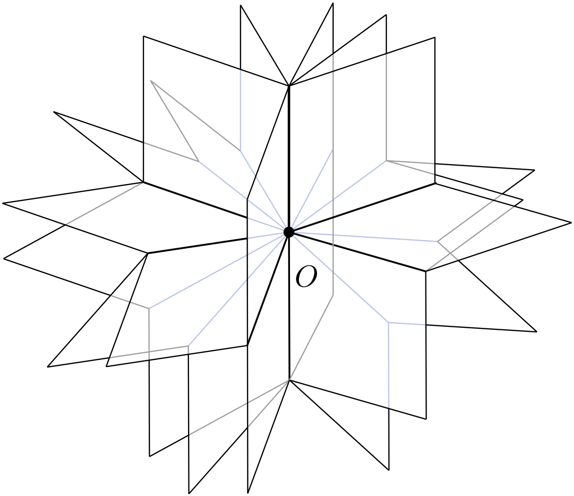
The space $\mathcal{T}_4$



Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)

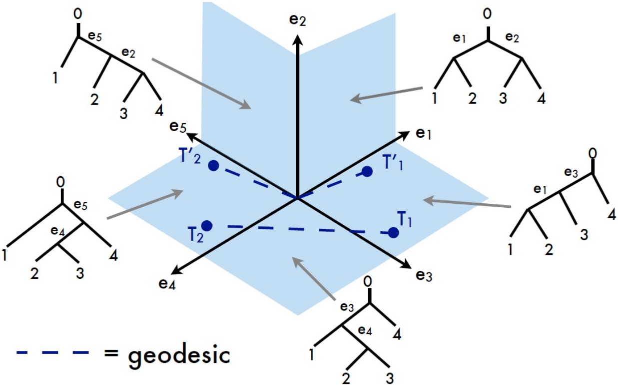


Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)



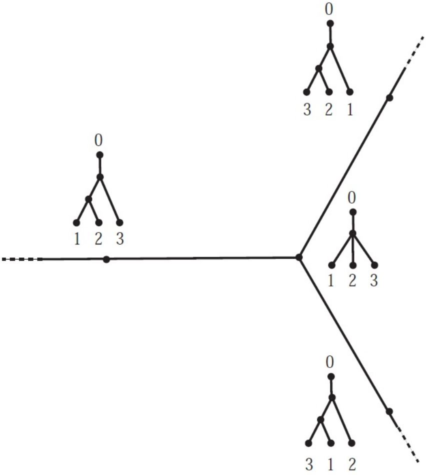
Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)

Geodesics in $\mathcal{T}_4$



Space of phylogenetic trees ([2001] Billera, Holmes, Vogtmann)

The space $\mathcal{T}_3$



Algorithmic problems in the space of phylogenetic trees

- [2001] L.J. Billera, S.P. Holmes and K. Vogtmann
  the space of phylogenetic trees

- [2011] M. Owen and S. Provan
  shortest path in the space of phylogenetic trees

- [2012] F. Ardila, M. Owen and S. Sullivant
  shortest path in CAT(0) cube complexes

Reconfigurable systems

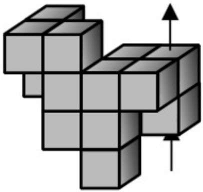
Reconfigurable robotic systems are composed of a set of robots that change their position relative to one another, thereby reshaping the system.

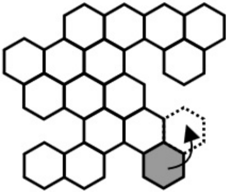
A robotic system that changes its shape due to individual robotic motion has been called **metamorphic**.

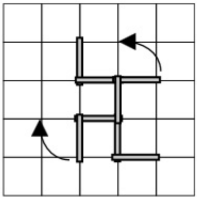
Reconfigurable systems

There are many models for such robots:

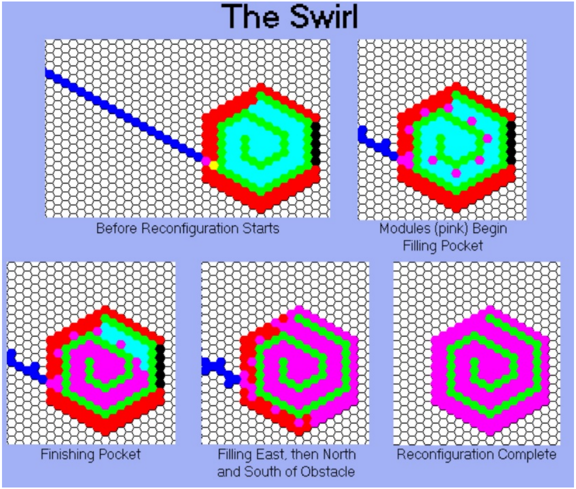
- 2D and 3D lattices;
- hexagonal, square, and dodecahedral cells;
- pivoting or sliding motion







Reconfigurable systems



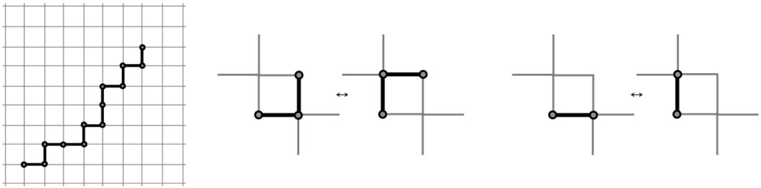
Reconfigurable systems ([2004] Abrams, Ghrist and Peterson)

Transition graph of the system - whose vertices are the states of the system and whose edges correspond to the allowable moves between them.

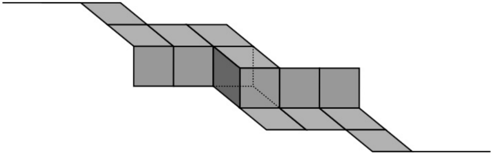
Abrams, Ghrist and Peterson observed that this graph is the 1-skeleton of the state complex: a cube complex whose vertices are the states of the system, whose edges correspond to allowable moves, and whose cubes correspond to collections of moves which can be performed simultaneously.

Reconfigurable systems ([2004] Abrams and Ghrist)

Beginning with a state having N vertical edges end-to-end, the metamorphic system thus generated models the position of an articulated robotic arm with fixed base which can (1) rotate at the top end and (2) flip corners as per the diagram.



Reconfigurable systems ([2004] Abrams and Ghrist)



Reconfigurable systems ([2013] Ardila, Baker and Yatchak)

Example 2.8. Figure 3 shows the state complex of the robot of 5 cells which starts horizontal in the lower right corner of a hexagonal tunnel of width 3, and is constrained to stay inside that tunnel. Notice that, due to the definition of the moves in Figure 1, the robot is not able to pivot to the top row or out of the lower right corner.

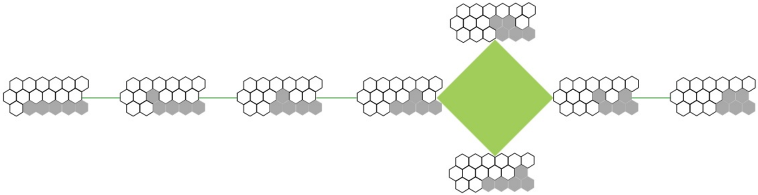


Figure 3: The state complex of a hexagonal metamorphic robot in a tunnel.

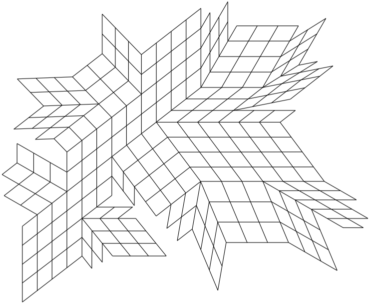
Related work on reconfigurable systems

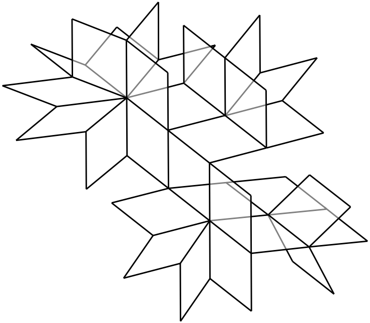
- [1991] V. Pratt
- [2004] A. Abrams and R. Ghrist
- [2007] R. Ghrist and V. Peterson
- [2013] F. Ardila, T. Baker and R. Yatchak

CAT(0) rectangular complexes

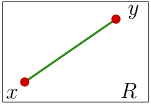
Definition

A **rectangular complex** $\mathcal{K}$ is a 2-dimensional Euclidean cell complex $\mathcal{K}$ whose 2-cells are isometric to axis-parallel rectangles of the l_1-plane and the intersection of two faces of $\mathcal{K}$ is either empty, either a vertex or an edge.



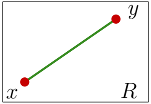


Intrinsic metric

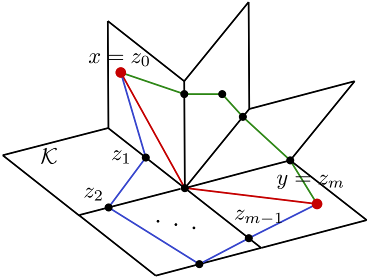


For all $x, y \in R$, $\boxed{d(x,y) = d_{\mathbb{E}^2}(x,y)}$

Intrinsic metric

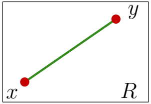


For all $x, y \in R$, $\boxed{d(x,y) = d_{\mathbb{E}^2}(x,y)}$

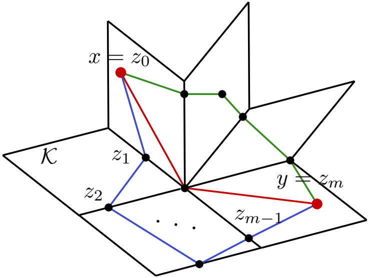


For all $x \in R'$ and $y \in R''$
Let $P = [x = z_0 z_1 \ldots z_{m-1} z_m = y]$,
$z_i, z_{i+1} \in R_i$, $i = 0, \ldots, m-1$.

Intrinsic metric



For all $x, y \in R$, $\boxed{d(x,y) = d_{\mathbb{E}^2}(x,y)}$



For all $x \in R'$ and $y \in R''$
Let $P = [x = z_0 z_1 \dots z_{m-1} z_m = y]$,
$z_i, z_{i+1} \in R_i$, $i = 0, \dots, m-1$.

$$\ell(P) = \sum_{i=0}^{m-1} d_{\mathbb{E}^2}(z_i, z_{i+1}),$$

$$\boxed{d(x,y) = \min_P \ell(P)}$$

CAT(0) cube complexes

Chepoi, 2000

CAT(0) cube complexes coincide with the cubical cell complexes arising from median graphs.

CAT(0) cube complexes

CAT(0) cube complexes can be described completely combinatorially.

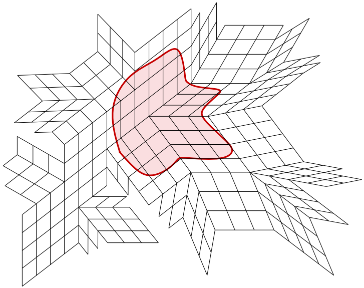
Gromov characterization

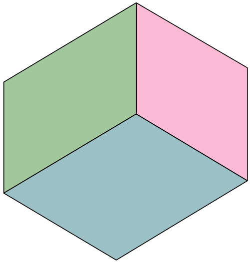
A cubical polyhedral complex $\mathcal{K}$ with the intrinsic metric is CAT(0) if and only if $\mathcal{K}$ is simply connected and satisfies the following condition: whenever three $(k+2)$-cubes of $\mathcal{K}$ share a common k-cube and pairwise share common $(k+1)$-cubes, they are contained in a $(k+3)$-cube of $\mathcal{K}$.

CAT(0) rectangular complexes

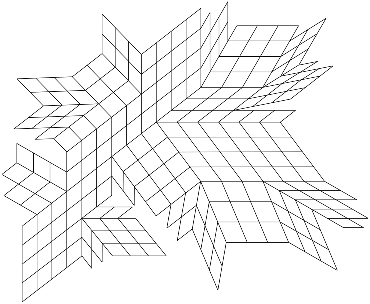
Gromov characterization

A rectangular complex $\mathcal{K}$ is a **CAT(0) rectangular complex** if and only if $\mathcal{K}$ is simply connected and $\mathcal{K}$ does not contain any triplet of faces pairwise adjacent.

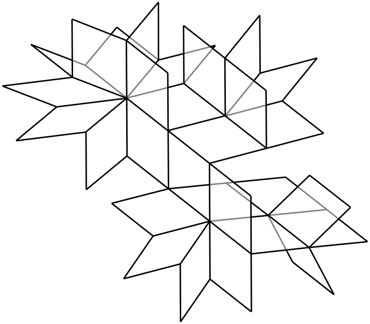




Examples of CAT(0) rectangular complexes



Squaregraphs

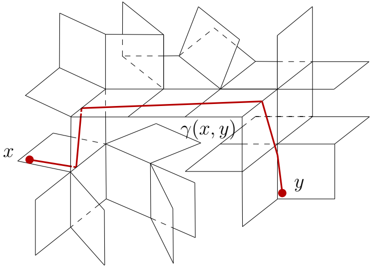


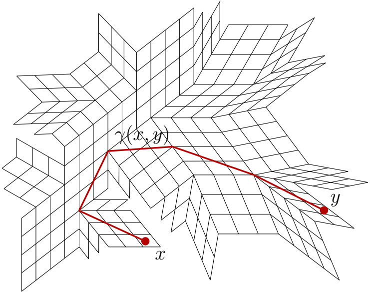
Ramified rectilinear polygons

Shortest path in a CAT(0) rectangular complex

SPP(x, y)

Given two points x, y in a CAT(0) rectangular complex $\mathcal{K}$, find the distance $d(x, y)$ and the unique shortest path $\gamma(x, y)$ connecting x and y in $\mathcal{K}$.

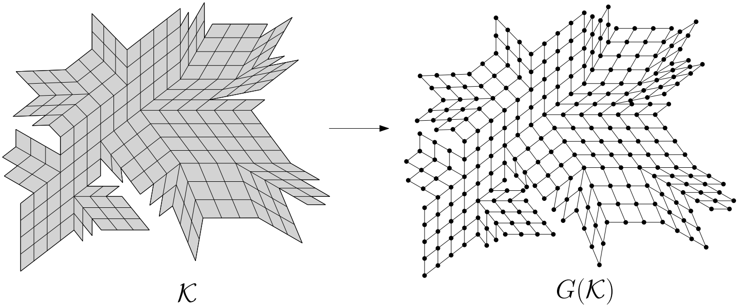




1-skeleton of a rectangular complex

Definition

The **1-skeleton** of a rectangular complex $\mathcal{K}$ is a graph $G(\mathcal{K}) = (V(\mathcal{K}), E(\mathcal{K}))$, where $V(\mathcal{K})$ is the vertex set of $\mathcal{K}$ and $E(\mathcal{K})$ is the edge set of $\mathcal{K}$.



Interval of two points

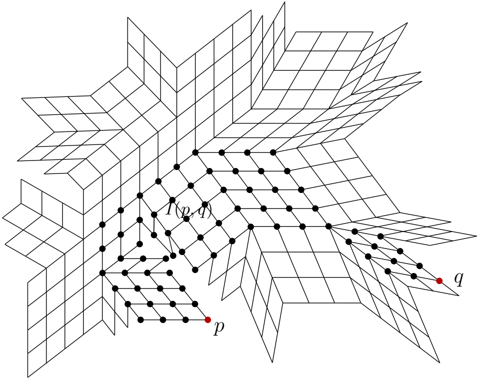
Definition

Given the vertices p and q in the 1-skeleton $G(\mathcal{K})$ of $\mathcal{K}$, the **interval** $I(p,q)$ is the set $\{z \in V(\mathcal{K}) : d_G(p,q) = d_G(p,z) + d_G(z,q)\}$.

Interval of two points

Definition

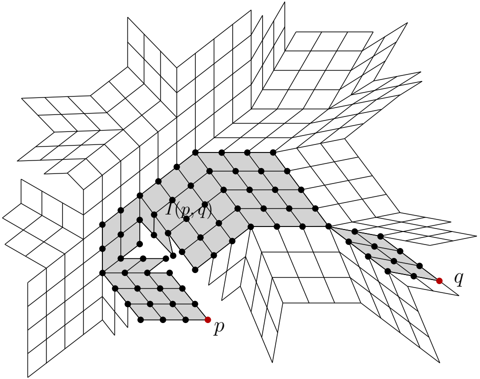
Given the vertices p and q in the 1-skeleton $G(\mathcal{K})$ of $\mathcal{K}$, the **interval** $I(p, q)$ is the set $\{z \in V(\mathcal{K}) : d_G(p, q) = d_G(p, z) + d_G(z, q)\}$.



Interval of two points

Definition

Given the vertices p and q in the 1-skeleton $G(\mathcal{K})$ of $\mathcal{K}$, the **interval** $I(p,q)$ is the set $\{z \in V(\mathcal{K}) : d_G(p,q) = d_G(p,z) + d_G(z,q)\}$.

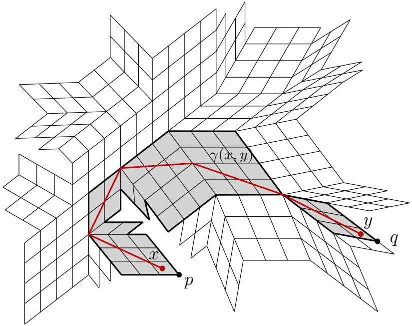


Let $\mathcal{K}(I(p,q))$ be the subcomplex of $\mathcal{K}$ induced by the interval $I(p,q)$.

Main result

Proposition 1

Given two points x and y in a CAT(0) rectangular complex $\mathcal{K}$, let R_x, R_y be the cells of $\mathcal{K}$ containing these points, then $\gamma(x, y) \subset \mathcal{K}(I(p, q))$, where p and q are two vertices of R_x and R_y.

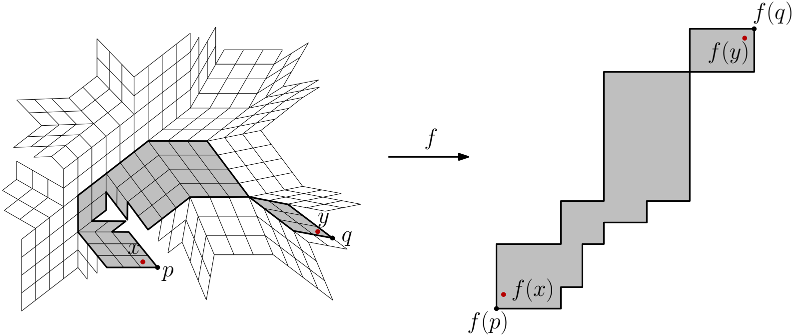


The same result is true for CAT(0) cubical complexes of any dimension.

Main result

Proposition 2

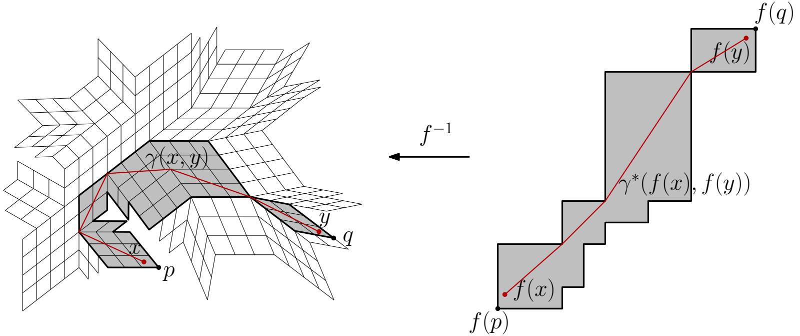
For each pair of points x, y of a CAT(0) rectangular complex $\mathcal{K}$, there exists an unfolding f of $\mathcal{K}(I(p,q))$ in the plane $\mathbb{R}^2$ as a chain of monotone polygons.
Moreover, $\gamma(x,y) = f^{-1}(\gamma^*(f(x), f(y)))$.



Main result

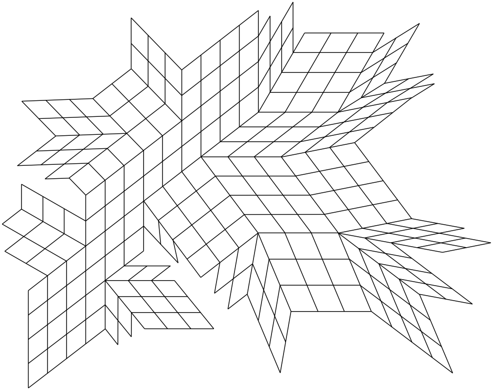
Proposition 2

For each pair of points x, y of a CAT(0) rectangular complex $\mathcal{K}$, there exists an unfolding f of $\mathcal{K}(I(p, q))$ in the plane $\mathbb{R}^2$ as a chain of monotone polygons. Moreover, $\gamma(x, y) = f^{-1}(\gamma^*(f(x), f(y)))$.



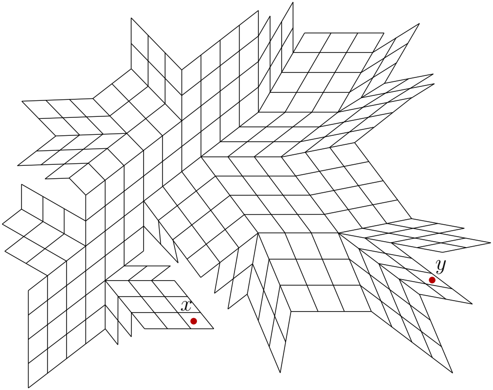
The main steps of the algorithm

1. Given the rectangular faces containing the points x and y, compute the vertices p, q of $\mathcal{K}$ such that $x, y \in \mathcal{K}(I(p, q))$.



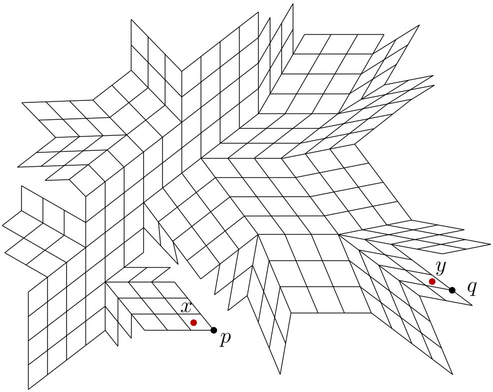
The main steps of the algorithm

1. Given the rectangular faces containing the points x and y, compute the vertices p, q of $\mathcal{K}$ such that $x, y \in \mathcal{K}(I(p, q))$.



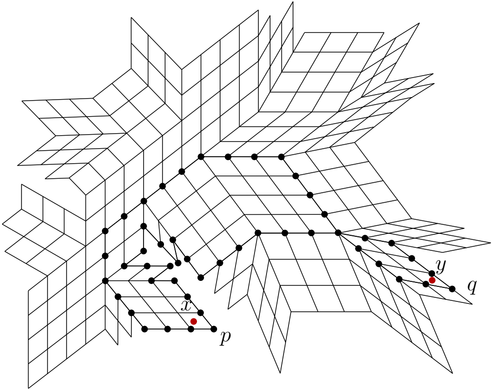
The main steps of the algorithm

1. Given the rectangular faces containing the points x and y, compute the vertices p, q of $\mathcal{K}$ such that $x, y \in \mathcal{K}(I(p, q))$.



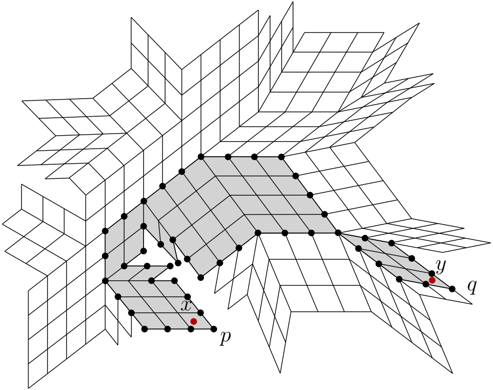
The main steps of the algorithm

2. Compute the boundary $\partial G(I(p,q))$.



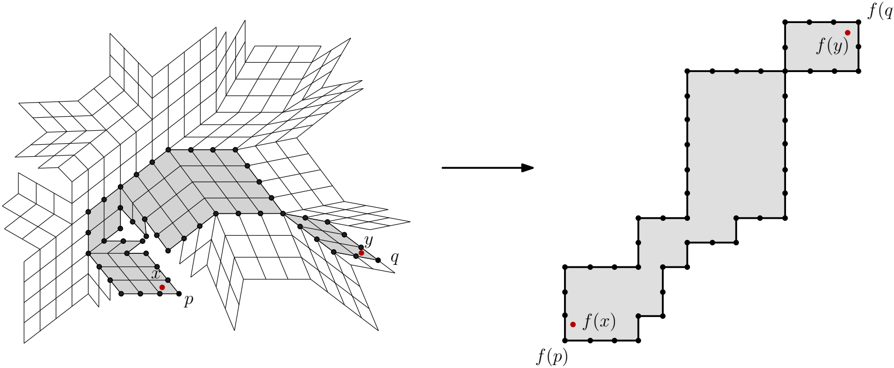
The main steps of the algorithm

2. Compute the boundary $\partial G(I(p, q))$.



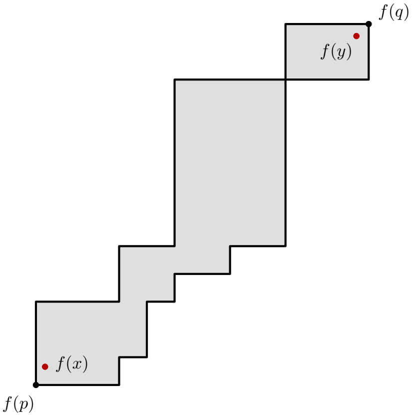
The main steps of the algorithm

3. Compute an unfolding f of $\partial G(I(p,q))$ in $\mathbb{R}^2$. Let $P(I(p,q))$ denote the chain of monotone polygons bounded by $f(\partial G(I(p,q)))$.



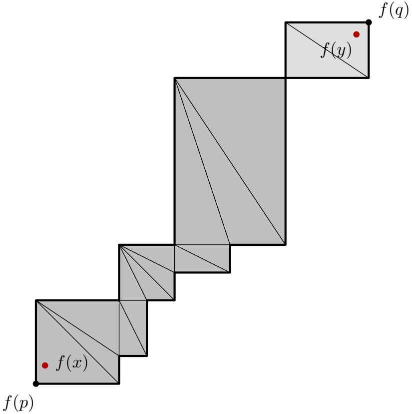
The main steps of the algorithm

3. Compute an unfolding f of $\partial G(I(p,q))$ in $\mathbb{R}^2$. Let $P(I(p,q))$ denote the chain of monotone polygons bounded by $f(\partial G(I(p,q)))$.



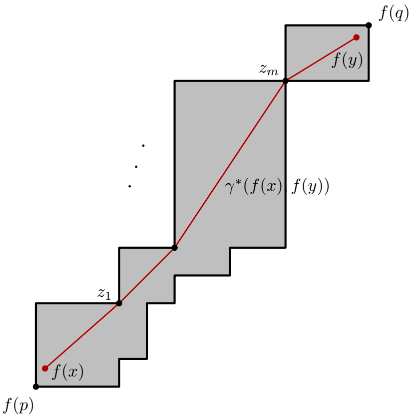
The main steps of the algorithm

4. Triangulate each monotone polygon constituting a block of $P(I(p,q))$.



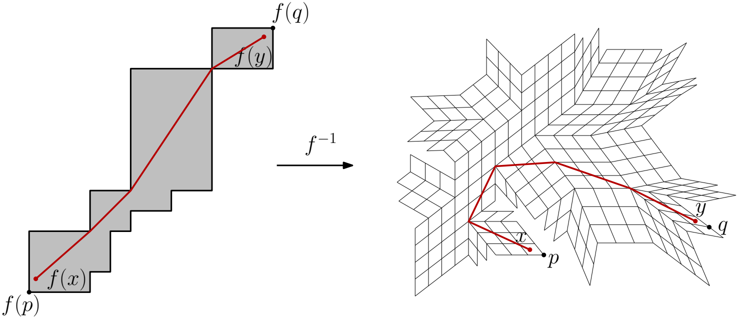
The main steps of the algorithm

5. In the triangulated polygon $P(I(p,q))$ compute the shortest path $\gamma^*(f(x), f(y)) = (f(x), z_1, \ldots, z_m, f(y))$ between $f(x)$ and $f(y)$ in $P(I(p,q))$, where $z_1, \ldots, z_m$ are all vertices of $P(I(p,q))$.



The main steps of the algorithm

6. Return $(x, f^{-1}(z_1), \dots, f^{-1}(z_m), y)$ as the shortest path $\gamma(x, y)$ between the points x and y.



The main steps of the algorithm

1. Compute the vertices p, q in $V(\mathcal{K})$ such that $x, y \in \mathcal{K}(I(p, q))$,
2. Construct the boundary $\partial G(I(p, q))$,
3. Find the unfolding f of $\partial G(I(p, q))$ in $\mathbb{R}^2$,
4. Compute the shortest path $\gamma^*(f(x), f(y))$,
5. Return $\gamma(x, y) := f^{-1}(\gamma^*(f(x), f(y)))$.

Main result

Theorem

Given a CAT(0) rectangular complex $\mathcal{K}$ with n vertices, one can construct a data structure $\mathcal{D}$ of size $O(n^2)$ so that, given any two points $x, y \in \mathcal{K}$, we can compute the shortest path $\gamma(x, y)$ between x and y in $O(d(p, q))$ time.

Open questions

- Design a subquadratic data structure D allowing to perform two-point shortest path queries in CAT(0) rectangular complexes in $O(d(p, q))$ time.

Open questions

- Design a subquadratic data structure D allowing to perform two-point shortest path queries in CAT(0) rectangular complexes in $O(d(p, q))$ time.

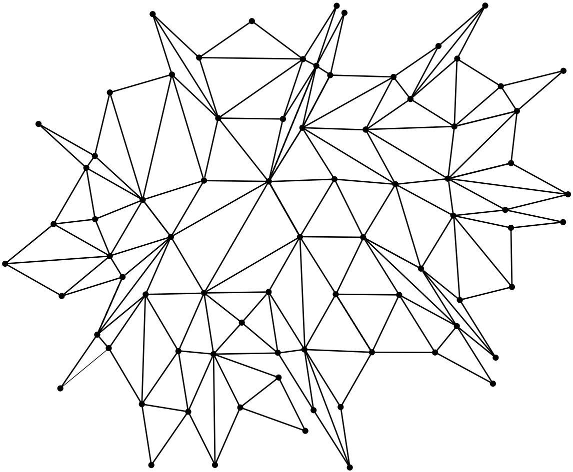
- Construct a polynomial algorithm for the two-point shortest path queries for all CAT(0) cubical complexes, in particular for 3-dimensional CAT(0) cubical complexes.

Open questions

- Computing geodesics in any CAT(0) cell complex.

Open questions

- Computing geodesics in any CAT(0) cell complex.

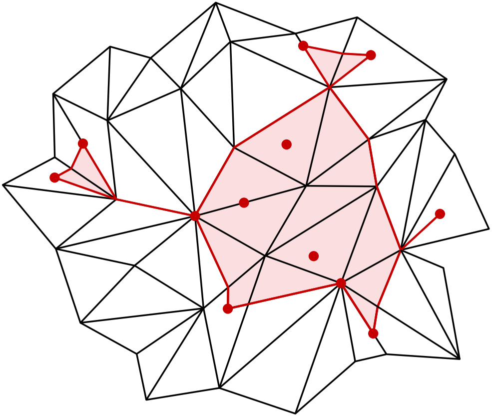


Open questions

- Constructing the convex hull of a finite set of points in a CAT(0) cell complex.

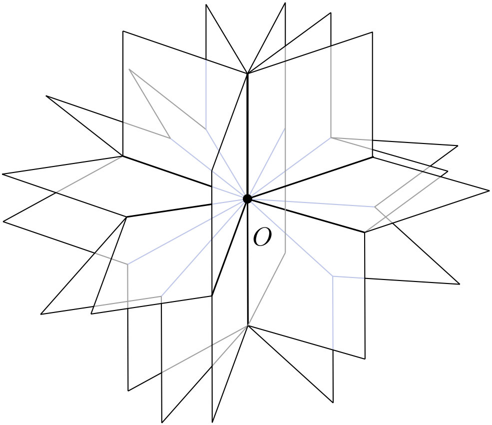
Open questions

- Constructing the convex hull of a finite set of points in a CAT(0) cell complex.



Open questions

- Constructing the convex hull of a finite set of points in a CAT(0) cell complex.



"A smile is the shortest distance between two people."

Victor Borge

Thank you!